THE LEFSCHETZ PROPERTY FOR BARYCENTRIC SUBDIVISIONS OF SHELLABLE COMPLEXES

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Abstract. We show that an 'almost strong Lefschetz' property holds for the barycentric subdivision of a shellable complex. From this we conclude that for the barycentric subdivision of a Cohen-Macaulay complex, the h-vector is unimodal, peaks in its middle degree (one of them if the dimension of the complex is even), and that its g-vector is an M-sequence. In particular, the (combinatorial) g-conjecture is verified for barycentric subdivisions of homology spheres. In addition, using the above algebraic result, we derive new inequalities on a refinement of the Eulerian statistics on permutations, where permutations are grouped by the number of descents and the image of 1.

1. Introduction

The starting point for this paper is Brenti and Welker's study of f-vectors of barycentric subdivisions of simplicial complexes [3]. They showed that for a Cohen-Macaulay complex, the h-vector of its barycentric subdivision is unimodal ([3, Corollary 3.5]). This raises the following natural questions about this h-vector: Where is its peak? Is the vector of its successive differences up to the middle degree ('g-vector') an M-sequence?

We answer these questions by finding an 'almost strong Lefschetz' element in case the original complex is shellable. Let us make this precise (for unexplained terminology see Section 2): let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex over a field k, on vertex set $[n] := \{1, \dots, n\}$ and $\Theta = \{\theta_1, ..., \theta_d\}$ a maximal linear system of parameters for its face ring $k[\Delta]$. We call a degree one element in the polynomial ring $\omega \in A = k[x_1, \dots, x_n]$ an *s-Lefschetz element for the A-module $k[\Delta]/\Theta$* if multiplication

$$\omega^{s-2i} : \ (k[\Delta]/\Theta)_i \ \longrightarrow \ (k[\Delta]/\Theta)_{s-i}$$

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is an injection for $0 \leq i \leq \lfloor \frac{s-1}{2} \rfloor$. A $(\dim \Delta)$-Lefschetz element is called an *almost strong Lefschetz element for* $k[\Delta]/\Theta$. (Recall that $k[\Delta]/\Theta$ is called strong Lefschetz if it has a $(\dim \Delta + 1)$-Lefschetz element.) Let $G_s(\Delta)$ be the set consisting of all pairs (Θ, ω) such that Θ is a maximal linear system of parameters for $k[\Delta]$ and ω is an s-Lefschetz element for $k[\Delta]/\Theta$. It can be shown that $G_s(\Delta)$ is a Zariski open set (e.g. imitate the proof in [14, Proposition 3.6]). If $G_s(\Delta) \neq \emptyset$ we say that Δ is *s-Lefschetz over k*, and that Δ is *almost strong Lefschetz over k* if $G_{\dim \Delta}(\Delta) \neq \emptyset$.

Theorem 1.1. *Let Δ be a shellable $(d-1)$-dimensional simplicial complex and let k be an infinite field. Then the barycentric subdivision of Δ is almost strong Lefschetz over k.*

This theorem has some immediate f-vector consequences; in particular it verifies the g-conjecture for barycentric subdivisions of homology spheres, and beyond. One of the main problems in algebraic combinatorics is the g-conjecture, first raised as question by McMullen for simplicial spheres [8]. Here we state the part of the conjecture which is still open.

Conjecture 1.2. (g-conjecture) Let L be a simplicial sphere, then its g-vector is an M-sequence.

It is conjectured to hold in greater generality, for all homology spheres and even for all doubly Cohen-Macaulay complexes, as was suggested by Björner and Swartz [14]. We verify these conjectures in a special case, as it was already conjectured in [2]:

Corollary 1.3. *Let Δ be a Cohen-Macaulay simplicial complex (over some field). Then the g-vector of the barycentric subdivision of Δ is an M-sequence. In particular, the g-conjecture holds for barycentric subdivisions of simplicial spheres, of homology spheres, and of doubly Cohen-Macaulay complexes.*

Note that the non-negativity of the g-vector of barycentric subdivisions of homology spheres already follows from Karu's result on the non-negativity of the cd-index for order complexes of Gorenstein* posets [6]. In Section 2 we provide some preliminaries and prove our main result Theorem 1.1. In Section 3 we derive some f-vector corollaries from Theorem 1.1, as well as extending this theorem to shellable polytopal complexes. In Section 4 we prove new inequalities for the refined Eulerian statistics on permutations, introduced by Brenti and Welker. The proofs are based on Theorem 1.1. As a corollary, the location of the peak of the h-vector of the barycentric subdivision of a Cohen-Macaulay complex is determined.

2. Proof of Theorem 1.1

Let Δ be a finite (abstract, non-empty) simplicial complex on a vertex set $\Delta_0 = [n] = \{1, 2, ..., n\}$, i.e. $\Delta \subseteq 2^{[n]}$ and if $S \subseteq T \in \Delta$ then $S \in \Delta$ (and $\emptyset \in \Delta$), and let Δ be of dimension $d-1$, i.e. $\max_{S\in\Delta} \#S = d$. The f-vector of Δ is $f^{\Delta} = (f_{-1}, f_0, ..., f_{d-1})$ where $f_{i-1} = \#\{S \in \Delta : \#S = i\}$. its h-vector, which carries the same combinatorial information, is $h^{\Delta} = (h_0, ..., h_d)$, where $\sum_{0\leq i\leq d} h_i x^{d-i} = \sum_{0\leq i\leq d} f_{i-1}(x-1)^{d-i}$.

Let k be an infinite field and $A = k[x_1, ..., x_n] = A_0 \oplus A_1 \oplus ...$ the polynomial ring graded by degree. The face ring (Stanley-Reisner ring) of Δ over k is $k[\Delta] = A/I_{\Delta}$ where I_{Δ} is the ideal $I_{\Delta} = (\prod_{1\leq i\leq n} x_i^{a_i} : \{i : a_i > 0\} \notin \Delta)$. It inherits the grading from A. If k is infinite, a maximal homogeneous system of parameters for $k[\Delta]$ can be chosen from the 'linear' part $k[\Delta]_1$, called l.s.o.p. for short. If $k[\Delta]$ is Cohen-Macaulay (CM for short) then any maximal l.s.o.p has d elements. We will denote such l.s.o.p. by $\Theta = \{\theta_1, ..., \theta_d\}$. In this case $k[\Delta]$ is a free $k[\Theta]$-module, and $h_i^{\Delta} = \dim_k(k[\Delta]/\Theta)_i$. We say that Δ is CM (over k) if $k[\Delta]$ is a CM ring.

The *barycentric subdivision* of a simplicial complex Δ is the simplicial complex $\mathrm{sd}(\Delta)$ on vertex set $\Delta \setminus \{\emptyset\}$ whose simplices are all the chains $F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_r$ of elements $F_i \in \Delta \setminus \{\emptyset\}$ for $0 \leq i \leq r$. The geometric realizations of Δ and $\mathrm{sd}(\Delta)$ are homeomorphic. Recall that Cohen-Macaulayness is a topological property. Hence, if Δ is CM then $\mathrm{sd}(\Delta)$ is CM as well. In this case Baclawski and Garsia ([1, Proposition 3.4.]) showed that $\Theta = \{\theta_1, \ldots, \theta_{\dim\Delta+1}\}$ where $\theta_i := \sum_{F\in\Delta, \#F=i} x_{\{F\}}$ for $1 \leq i \leq \dim\Delta + 1$, is a l.s.o.p. for $k[\mathrm{sd}(\Delta)]$. For further terminology and background we refer to Stanley's book [13]. Let us start with some auxiliary results.

We denote by $\mathrm{cone}(\Delta)$ the cone over Δ, i.e. $\mathrm{cone}(\Delta)$ is the join of some vertex $\{v\}$ with Δ where $v \notin \Delta$, $\mathrm{cone}(\Delta) := \{\ F \mid F \in \Delta\} \cup \{\ F \cup \{v\} \mid F \in \Delta\}$. The following lemma deals with the effect of coning on the s-Lefschetz property.

Lemma 2.1. *Let Δ be a $(d-1)$-dimensional simplicial complex. If Δ is s-Lefschetz over k then the same is true for* $\mathrm{cone}(\Delta)$.

Proof. Let Θ be a l.s.o.p. for $k[\Delta]$ and let v be the apex of $\mathrm{cone}(\Delta)$. Then $\widetilde{\Theta} := \Theta \cup \{x_v\}$ is a l.s.o.p. for $k[\mathrm{cone}(\Delta)]$ (to see this, use the isomorphism $k[\mathrm{cone}(\Delta)] \cong k[\Delta] \otimes_k k[x_v]$ of modules over $k[x_u : u \in \Delta_0 \cup \{v\}] \cong k[x_u : u \in \Delta_0] \otimes_k k[x_v]$). Furthermore, $k[\Delta]/\Theta \cong k[\mathrm{cone}(\Delta)]/\widetilde{\Theta}$ as A-modules, where $A = k[x_i : i \in \{v\} \cup \Delta_0]$ and $x_v \cdot k[\Delta] = 0$. Hence,

for any pair $(\Theta, \omega) \in G_s(\Delta)$ we have $(\widetilde{\Theta}, \omega) \in G_s(\mathrm{cone}(\Delta))$, and the assertion follows. $\square$

Note that if Δ is almost strong Lefschetz over k then $\mathrm{cone}(\Delta)$ is $(\dim \Delta)$-Lefschetz over k.

The following theorem is the main part of Stanley's proof of the necessity part of the g-theorem for simplicial polytopes [12].

Theorem 2.2 ([12]). *Let P be a simplicial d-polytope and let Δ be the boundary complex of P. Then Δ is d-Lefschetz over $\mathbb{R}$.*

If Δ is a simplicial complex and $\{F_1, \ldots, F_m\} \subseteq \Delta$ is a collection of faces of Δ we denote by $\langle F_1, \ldots, F_m \rangle$ the simplicial complex whose faces are the subsets of the F_i's, $1 \leq i \leq m$.

For an arbitrary infinite field k (of arbitrary characteristic!) the conclusion in Theorem 2.2 holds for the following polytopes, which will suffice for concluding our main result Theorem 1.1:

Proposition 2.3. *Let P be a d-simplex and let Δ be its barycentric subdivision. Let k be an infinite field. Then Δ is almost strong Lefschetz over k.*

Proof. Note that the boundary complex $\partial\Delta$ is obtained from ∂P by a sequence of stellar subdivisions - order the faces of ∂P by decreasing dimension and perform a stellar subdivision at each of them according to this order to obtain $\partial\Delta$. In particular, $\partial\Delta$ is *strongly edge decomposable*, introduced in [10], as the inverse stellar moves when going backwards in this sequence of complexes demonstrate.

It was shown by Murai [9, Corollary 3.5] that strongly edge decomposable complexes have the strong Lefschetz property (see also [11, Corollary 4.6.6]). As $\Delta = \mathrm{cone}(\partial\Delta)$, we conclude that Δ is d-Lefschetz over k by Lemma 2.1. $\square$

We would like to point out that the proof of Proposition 2.3 is self-contained and does not require Theorem 2.2. Shellability of simplicial complexes is a useful tool in combinatorics; here we give two equivalent definitions for shellability which we will use later.

Definition 2.4. A pure simplicial complex Δ is called *shellable* if Δ is a simplex or if one of the following equivalent conditions is satisfied. There exists a linear ordering $F_1, \ldots, F_m$ of the facets of Δ such that

(a) $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a non-empty set of maximal proper faces of $\langle F_i \rangle$, for all $2 \leq i \leq m$.
(b) the set $\{F \mid F \in \langle F_1, \ldots, F_i \rangle, F \notin \langle F_1, \ldots, F_{i-1} \rangle\}$ has a unique minimal element for all $2 \leq i \leq m$. This element is called the *restriction face* of F_i. We denote it by $\mathrm{res}(F_i)$.

A linear order of the facets satisfying the equivalent conditions (a) and (b) is called a *shelling* of Δ.

We are now in position to prove Theorem 1.1. In the sequel, we will loosely use the term 'generic elements' to mean that these elements are chosen from a Zariski non-empty open set, to be understood from the context.

Proof of Theorem 1.1. The proof is by double induction, on the number of facets $f^{\Delta}_{\dim\Delta}$ of Δ and on the dimension of Δ. Let $\dim\Delta \geq 0$ be arbitrary and $f^{\Delta}_{\dim\Delta} = 1$, i.e. Δ is a $(d-1)$-simplex, and by Proposition 2.3 we are done. Let $\dim\Delta = 0$, i.e. Δ as well as $\mathrm{sd}(\Delta)$ consist of vertices only. Since $h^{\mathrm{sd}(\Delta)}_0 = h^{\mathrm{sd}(\Delta)}_{1-1-0}$ there is nothing to show. This provides the base of the induction.

For the induction step let $\dim\Delta \geq 1$. Let $n = f^{\mathrm{sd}(\Delta)}_0$ and let $A = k[x_1,\ldots,x_n]$ be the polynomial ring in n variables. Let $F_1,\ldots,F_m$ be a shelling of Δ with $m \geq 2$ and let $\widetilde{\Delta} := \langle F_1,\ldots,F_{m-1}\rangle$. Then $\sigma := \widetilde{\Delta} \cap \langle F_m\rangle$ is a pure $(d-2)$-dimensional subcomplex of ∂F_m. The barycentric subdivision $\mathrm{sd}(\Delta)$ of Δ is given by $\mathrm{sd}(\Delta) = \mathrm{sd}(\widetilde{\Delta}) \cup \mathrm{sd}(\langle F_m\rangle)$ and $\mathrm{sd}(\sigma) = \mathrm{sd}(\widetilde{\Delta}) \cap \mathrm{sd}(\langle F_m\rangle)$.

We get the following Mayer-Vietoris exact sequence of A-modules:

$$0 \to k[\mathrm{sd}(\Delta)] \to k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(\langle F_m\rangle)] \to k[\mathrm{sd}(\sigma)] \to 0. \tag{1}$$

Here the injection on the left-hand side is given by $\alpha \mapsto (\tilde{\alpha}, -\tilde{\alpha})$ and the surjection on the right-hand side by $(\beta,\gamma) \mapsto \tilde{\beta}+\tilde{\gamma}$, where $\tilde{a}$ denotes the obvious projection of a on the appropriate quotient module. (For a subcomplex Γ of Δ and $v \in \Delta_0 \setminus \Gamma_0$ it holds that $x_v \cdot k[\Gamma] = 0$.)

Let $\Theta = \{\theta_1,\ldots,\theta_d\}$ be a (maximal) l.s.o.p. for $k[\mathrm{sd}(\Delta)]$, $k[\mathrm{sd}(\widetilde{\Delta})]$ and $k[\mathrm{sd}(F_m)]$, and such that $\{\theta_1,\ldots,\theta_{d-1}\}$ is a l.s.o.p. for $k[\sigma]$. This is possible, as the intersection of finitely many non-empty Zariski open sets is non-empty (for $k[\sigma]$, its set of maximal l.s.o.p.'s times k^n (for θ_d) is Zariski open in k^{dn}). Dividing out by Θ in the short exact sequence (1), which is equivalent to tensoring with $-\otimes_A A/\Theta$, yields the following Tor-long exact sequence:

$$\begin{aligned}
\ldots\ &\to \mathrm{Tor}_1(k[\mathrm{sd}(\Delta)], A/\Theta) \to \mathrm{Tor}_1(k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(\langle F_m\rangle)], A/\Theta) \\
\to\ &\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta) \xrightarrow{\delta} \mathrm{Tor}_0(k[\mathrm{sd}(\Delta)], A/\Theta) \\
\to\ &\mathrm{Tor}_0(k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(\langle F_m\rangle)], A/\Theta) \to \mathrm{Tor}_0(k[\mathrm{sd}(\sigma)], A/\Theta) \to 0,
\end{aligned}$$

where $\delta: \ \mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta) \to \mathrm{Tor}_0(k[\mathrm{sd}(\Delta)], A/\Theta)$ is the connecting homomorphism. Below we write $k(\mathrm{sd}(\Delta))$ for $k[\mathrm{sd}(\Delta)]/\Theta$, and similarly $k(\mathrm{sd}(\widetilde{\Delta}))$, $k(\mathrm{sd}(\sigma))$ and $k(\mathrm{sd}(\langle F_m\rangle))$ for $k[\mathrm{sd}(\widetilde{\Delta})]/\Theta$, $k[\mathrm{sd}(\sigma)]/\Theta$ and $k[\mathrm{sd}(\langle F_m\rangle)]/\Theta$ resp.

Using that for R-modules M, N and Q it holds that $\mathrm{Tor}_0(M,N) \cong M \otimes_R N$, $(M \oplus N) \otimes_R Q \cong (M \otimes_R Q) \oplus (N \otimes_R Q)$ and that $M/IM \cong M \otimes_R R/I$ for an ideal $I \lhd R$, we get the following exact sequence of A-modules:

$$\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta) \xrightarrow{\delta} k(\mathrm{sd}(\Delta)) \to k(\mathrm{sd}(\widetilde{\Delta})) \oplus k(\mathrm{sd}(\langle F_m\rangle)) \to k(\mathrm{sd}(\sigma)) \to 0.$$

Note that all the maps in this sequence are grading preserving, where $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta)$ inherits the grading from (a projective grading preserving resolution of) the sequence (1). From this we deduce the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Tor}_1(k(\mathrm{sd}(\sigma)))_i & \xrightarrow{\delta} & k(\mathrm{sd}(\Delta))_i & \to & k(\mathrm{sd}(\widetilde{\Delta}))_i \oplus k(\mathrm{sd}(\langle F_m\rangle))_i \\ & & \downarrow \omega^{d-2i-1} & & \downarrow (\omega^{d-2i-1}, \omega^{d-2i-1}) \\ & & k(\mathrm{sd}(\Delta))_{d-1-i} & \to & k(\mathrm{sd}(\widetilde{\Delta}))_{d-1-i} \oplus k(\mathrm{sd}(\langle F_m\rangle))_{d-1-i} \end{array}$$

where ω is a degree one element in A. Since F_m is a $(d-1)$-simplex we know from the base of the induction that multiplication

$$\omega^{d-2i-1}: \ k(\mathrm{sd}(\langle F_m\rangle))_i \to k(\mathrm{sd}(\langle F_m\rangle))_{d-1-i}$$

is an injection for a generic choice of ω in A_1. (Note that if G is a Zariski open set in $k[x_v : v \in (F_m)_0]_1$ then $G \times k[x_v : v \in \Delta_0 \setminus (F_m)_0]_1$ is Zariski open in A_1.)

By construction, $\widetilde{\Delta}$ is shellable and therefore by the induction hypothesis the multiplication

$$\omega^{d-2i-1}: \ k(\mathrm{sd}(\widetilde{\Delta}))_i \to k(\mathrm{sd}(\widetilde{\Delta}))_{d-1-i}$$

is an injection for generic ω. Since the intersection of two non-empty Zariski open sets is non-empty, multiplication

$$(\omega^{d-2i-1}, \omega^{d-2i-1}) : k(\mathrm{sd}(\widetilde{\Delta}))_i \oplus k(\mathrm{sd}(\langle F_m\rangle))_i \to k(\mathrm{sd}(\widetilde{\Delta}))_{d-1-i} \oplus k(\mathrm{sd}(\langle F_m\rangle))_{d-1-i}$$

is an injection for a generic $\omega \in A_1$.

Our aim is to show that $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta)_i = 0$ for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$. As soon as this is shown, the above commutative diagram implies that multiplication

$$\omega^{d-2i-1}: \ k(\mathrm{sd}(\Delta))_i \to k(\mathrm{sd}(\Delta))_{d-1-i}$$

is injective for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$ and ω as above.

For the computation of $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta)$ we consider the following exact sequence of A-modules:

$$0 \to \Theta A \to A \to A/\Theta \to 0.$$

Since $\mathrm{Tor}_0(M, N) \cong M \otimes_R N$ and $\mathrm{Tor}_1(R, M) = 0$ for R-modules M and N, we get the following Tor-long exact sequence

$$0 \to \mathrm{Tor}_1(A/\Theta, k[\mathrm{sd}(\sigma)]) \to \Theta A \otimes_A k[\mathrm{sd}(\sigma)] \to k[\mathrm{sd}(\sigma)] \to A/\Theta \otimes_A k[\mathrm{sd}(\sigma)] \to 0.$$

From the exactness of this sequence we deduce $\mathrm{Tor}_1(A/\Theta, k[\mathrm{sd}(\sigma)]) = \mathrm{Ker}(\Theta A \otimes_A k[\mathrm{sd}(\sigma)] \to k[\mathrm{sd}(\sigma)])$. Since we have $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta) \cong \mathrm{Tor}_1(A/\Theta, k[\mathrm{sd}(\sigma)])$, and by the fact that the isomorphism is grading preserving, we finally get that $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta) \cong \mathrm{Ker}(\Theta A \otimes_A k[\mathrm{sd}(\sigma)] \to k[\mathrm{sd}(\sigma)])$ as graded A-modules. The grading of $\Theta A \otimes_A k[\mathrm{sd}(\sigma)]$ is given by $deg(f \otimes_A g) = deg_A(f) + deg_A(g)$, where deg_A refers to the grading induced by A.

As mentioned before, for generic Θ, $\widetilde{\Theta} := \{\theta_1, \dots, \theta_{d-1}\}$ is a l.s.o.p. for $k[\mathrm{sd}(\sigma)]$. Thus the kernel of the map

$$(\Theta A \otimes_A k[\mathrm{sd}(\sigma)])_i \to (k[\mathrm{sd}(\sigma)])_i;\ \ b \otimes f \mapsto bf$$

is zero iff the kernel of the map

$$((\theta_d) \otimes_A (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta}))_i \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i;\ \ \theta_d \otimes f \mapsto \theta_d f$$

is zero, which is the case iff the kernel of the multiplication map

$$\theta_d :\ \ (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_{i-1} \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i;\ \ f \mapsto \theta_d f$$

is zero. (We have a shift (-1) in the grading since the last map θ_d increases the degree by $+1$).

By construction, σ is a pure subcomplex of the boundary of a $(d-1)$-simplex and thus is shellable. Since $\dim(\sigma) = d-2$ the induction hypothesis applies to $\mathrm{sd}(\sigma)$. Thus, multiplication

$$\theta_d^{d-2i-2} :\ \ (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_{d-i-2}$$

is an injection for $0 \le i \le \lfloor \frac{d-3}{2} \rfloor$ for a generic degree one element θ_d. In particular, multiplication

$$\theta_d :\ \ (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_{i+1}$$

is injective as well. Thus, $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta)_i = 0$ for $1 \le i \le \lfloor \frac{d-3}{2} \rfloor + 1 = \lfloor \frac{d-1}{2} \rfloor$. In particular, $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta)_i = 0$ for $1 \le i \le \lfloor \frac{d-2}{2} \rfloor$. Note that $(\Theta A \otimes_A k[\mathrm{sd}(\sigma)])_0 = 0$, hence $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta)_0 = 0$. To summarize, $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], A/\Theta)_i = 0$ for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$, which completes the proof. $\square$

3. Combinatorial Consequences

We are now going to discuss some combinatorial consequences of Theorem 1.1.

For a $(d-1)$-dimensional simplicial complex Δ let its g-vector be $g^{\Delta} := (g_0^{\Delta}, g_1^{\Delta}, \ldots, g_{\lfloor \frac{d}{2} \rfloor}^{\Delta})$, where $g_0^{\Delta} = 1$ and $g_i^{\Delta} = h_i^{\Delta} - h_{i-1}^{\Delta}$ for $1 \le i \le \lfloor \frac{d}{2} \rfloor$. A sequence $(a_0, \ldots, a_t)$ is called an M-sequence if it is the Hilbert function of a standard graded Artinian k-algebra. Macaulay [7] gave a characterization of such sequences by means of numerical conditions among their elements (see e.g. [13]).

Recall that shellable complexes are CM (e.g. [4]). While the converse is not true, Stanley showed that these two families of complexes have the same set of h-vectors:

Theorem 3.1 (Theorem 3.3 [13])**.** *Let $s = (h_0, \ldots, h_d)$ be a sequence of integers. The following conditions are equivalent:*

(i) *s is the h-vector of a shellable simplicial complex.*
(ii) *s is the h-vector of a Cohen-Macaulay simplicial complex.*
(iii) *s is an M-sequence.*

We are now going to prove some f-vector corollaries of Theorem 1.1, using Theorem 3.1 and Theorem 4.2.

Proof of Corollary 1.3. For any simplicial complex Γ, $h^{\mathrm{sd}(\Gamma)}$ is a function of h^{Γ}. (For an explicit formula, see Theorem 4.2 below, obtained in [3].) Hence, together with Theorem 3.1 we can assume that Δ is shellable. Let $\dim \Delta = d-1$.

By Theorem 1.1, for a generic l.s.o.p. Θ and a generic degree one element ω, multiplication

$$\omega^{d-1-2i} : \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)]/\Theta)_{d-1-i}$$

is an injection for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$, hence

$$\omega : \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)]/\Theta)_{i+1}$$

is an injection as well. (This conclusion is vacuous for $d \le 1$.) Therefore, as Cohen-Macaulayness implies $h_i^{\mathrm{sd}(\Delta)} = \dim_k(k[\mathrm{sd}(\Delta)]/\Theta)_i$, we get that $g_i^{\mathrm{sd}(\Delta)} = \dim_k(k[\mathrm{sd}(\Delta)]/(\Theta, \omega))_i$ for $0 \le i \le \lfloor \frac{d}{2} \rfloor$. Hence $g^{\mathrm{sd}(\Delta)}$ is an M-sequence. □

Corollary 3.2. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex. Then $h_{d-i-1}^{\mathrm{sd}(\Delta)} \ge h_i^{\mathrm{sd}(\Delta)}$ for any $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$.*

Proof. Again, by Theorems 4.2 and 3.1 we can assume that Δ is shellable. By Theorem 1.1, for a generic l.s.o.p. Θ of $k[\mathrm{sd}(\Delta)]$ multiplication

$$\omega^{d-1-2i}: \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)]/\Theta)_{d-1-i}$$

is an injection for $1 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$ and a generic degree one element ω. Since $h_i^{\mathrm{sd}(\Delta)} = \dim_k(k[\mathrm{sd}(\Delta)]/\Theta)_i$ this implies $h_i^{\mathrm{sd}(\Delta)} \leq h_{d-1-i}^{\mathrm{sd}(\Delta)}$. □

Next we verify the almost strong Lefschetz property for polytopal complexes. The proof essentially follows the same steps as the one of Theorem 1.1; we will indicate the differences.

A *polytopal complex* is a finite, non-empty collection $\mathcal{C}$ of polytopes (called the faces of $\mathcal{C}$) in some $\mathbb{R}^t$ that contains all the faces of its polytopes, and such that the intersection of two of its polytopes is a face of each of them. Notions like facets, dimension, pureness and barycentric subdivision are defined as usual.

Shellability extends to polytopal complexes as follows (see e.g. [15] for more details).

Definition 3.3. Let $\mathcal{C}$ be a pure $(d-1)$-dimensional polytopal complex. A *shelling* of $\mathcal{C}$ is a linear ordering $F_1, F_2, \ldots, F_m$ of the facets of $\mathcal{C}$ such that either $\mathcal{C}$ is 0-dimensional, or it satisfies the following conditions:

(i) The boundary complex $\mathcal{C}(\partial F_1)$ of the first facet F_1 has a shelling.
(ii) For $1 < j \leq m$ the intersection of the facet F_j with the previous facets is non-empty and is the beginning segment of a shelling of the $(d-2)$-dimensional boundary complex of F_j, that is,

$$F_j \cap (\bigcup_{i=1}^{j-1} F_i) = G_1 \cup G_2 \cup \ldots \cup G_r$$

for some shelling $G_1, G_2, \ldots, G_r, \ldots, G_t$ of $\mathcal{C}(\partial F_j)$, and $1 \leq r \leq t$.

A polytopal complex is called *shellable* if it is pure and has a shelling.

Theorem 3.4. *Let Δ be a shellable $(d-1)$-dimensional polytopal complex. Then $\mathrm{sd}(\Delta)$ is almost strong Lefschetz over $\mathbb{R}$. In particular, $g^{\mathrm{sd}(\Delta)}$ is an M-sequence.*

Proof. We give a sketch of the proof, indicating the needed modifications w.r.t. the proof of Theorem 1.1.

We use induction on the dimension and the number of facets f_{d-1}^{Δ} of Δ. For $f_{d-1}^{\Delta} = 1$, note that the barycentric subdivision of a polytope is combinatorially isomorphic to a simplicial polytope (see [5]).

Theorem 2.2 implies that $\mathrm{sd}(\partial P)$ is $(d-1)$-Lefschetz over $\mathbb{R}$. By Lemma 2.1 the same holds for $\mathrm{cone}(\mathrm{sd}(\partial P)) = \mathrm{sd}(P)$. Together with the $\dim \Delta = 0$ case, this provides the base of the induction.

The induction step works as in the proof of Theorem 1.1. □

Note that in the above proof we really need the classical g-theorem, whereas in the proof of Theorem 1.1 it was not required.

4. New Inequalities for the refined Eulerian statistics on permutations

In [3] Brenti and Welker give a precise description of the h-vector of the barycentric subdivision of a simplicial complex in terms of the h-vector of the original complex. The coefficients that occur in this representation are a refinement of the Eulerian statistics on permutations.

Let S_d denotes the symmetric group on $[d]$, and let $\sigma \in S_d$. We write $D(\sigma) := \{\ i \in [d-1] \mid \sigma(i) > \sigma(i+1)\}$ for the descent set of σ and $\mathrm{des}(\sigma) := \#D(\sigma)$ counts the number of descents of σ. For $0 \leq i \leq d-1$ and $1 \leq j \leq d$ we set $A(d,i,j) := \#\{\ \sigma \in S_d \mid des(\sigma) = i,\ \sigma(1) = j\}$. Brenti and Welker showed that these numbers satisfy the following symmetry:

Lemma 4.1. [3, Lemma 2.5]

$$A(d,i,j) = A(d,d-1-i,d+1-j)$$

for $d \geq 1$, $1 \leq j \leq d$ and $0 \leq i \leq d-1$.

The following theorem establishes the relation between the h-vector of a simplicial complex and the h-vector of its barycentric subdivision. As stated in [3] the result actually holds in the generality of Boolean cell complexes.

Theorem 4.2. [3, Theorem 2.2] *Let Δ be a $(d-1)$-dimensional simplicial complex and let $\mathrm{sd}(\Delta)$ be its barycentric subdivision. Then*

$$h_j^{\mathrm{sd}(\Delta)} = \sum_{r=0}^{d} A(d+1,j,r+1)h_r^{\Delta}$$

for $0 \leq j \leq d$.

In order to prove some new inequalities for the $A(d,i,j)$'s we will need the following characterization of the h-vector of a shellable simplicial complex due to McMullen and Walkup.

Proposition 4.3. [4, Corollary 5.1.14] *Let Δ be a shellable $(d-1)$-dimensional simplicial complex with shelling $F_1, \ldots, F_m$. For $2 \le j \le m$, let r_j be the number of facets of $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ and set $r_1 = 0$. Then $h_i^{\Delta} = \#\{ j \mid r_j = i\}$ for $i = 0, \ldots, d$. In particular, the numbers h_j^{Δ} do not depend on the particular shelling.*

It is easily seen that $r_j = \#\mathrm{res}(F_j)$. We will use this fact in the proof of the following corollary.

Corollary 4.4. (i) *$A(d+1, j, r) \le A(d+1, d-1-j, r)$ for $d \ge 0$, $1 \le r \le d+1$ and $0 \le j \le \lfloor \frac{d-2}{2} \rfloor$.*

(ii)

$$A(d+1, 0, r+1) \le A(d+1, 1, r+1) \le \ldots \le A(d+1, \lfloor \frac{d}{2} \rfloor, r+1)$$

and

$$A(d+1, d, r+1) \le A(d+1, d-1, r+1) \le \ldots \le A(d+1, \lceil \frac{d}{2} \rceil, r+1)$$

for $d \ge 1$ and $1 \le r \le d$.(For d odd, $A(d+1, \lfloor \frac{d}{2} \rfloor, r+1)$ may be larger or smaller then $A(d+1, \lceil \frac{d}{2} \rceil, r+1)$.)

Proof. Let Δ be a shellable $(d-1)$-dimensional simplicial complex. Let $F_1, \ldots, F_m$ be a shelling of Δ with $m \ge 2$ and set $\widetilde{\Delta} := \langle F_1, \ldots, F_{m-1} \rangle$. Since $\mathrm{sd}(\widetilde{\Delta})$ is a subcomplex of $\mathrm{sd}(\Delta)$ we get the following short exact sequence of A-modules for $A = k[x_1, \ldots, x_{f_0^{\mathrm{sd}(\Delta)}}]$:

$$0 \to I \to k[\mathrm{sd}(\Delta)] \to k[\mathrm{sd}(\widetilde{\Delta})] \to 0,$$

where I denotes the kernel of the projection on the right-hand side. Let Θ be a maximal l.s.o.p. for both $k[\mathrm{sd}(\Delta)]$ and $k[\mathrm{sd}(\widetilde{\Delta})]$. As $\widetilde{\Delta}$ is shellable it is CM and therefore $\mathrm{sd}(\widetilde{\Delta})$ is CM as well. Hence dividing out by Θ yields the following exact sequence of A-modules:

$$0 \to I/(I \cap \Theta) \to k[\mathrm{sd}(\Delta)]/\Theta \to k[\mathrm{sd}(\widetilde{\Delta})]/\Theta \to 0. \quad (2)$$

Consider the following commutative diagram

$$\begin{array}{ccccc} 0 & \to & I/(I \cap \Theta)_i & \to & (k[\mathrm{sd}(\Delta)]/\Theta)_i \\ & & \downarrow \omega^{d-1-2i} & & \downarrow \omega^{d-1-2i} \\ 0 & \to & I/(I \cap \Theta)_{d-1-i} & \to & (k[\mathrm{sd}(\Delta)]/\Theta)_{d-1-i} \end{array}$$

where ω is in A_1. By Theorem 1.1 multiplication

$$\omega^{d-2i-1}: \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)])_{d-1-i}$$

is an injection for $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$ and generic ω. It hence follows that also multiplication

$$\omega^{d-1-2i}: \ (I/(I \cap \Theta))_i \to (I/(I \cap \Theta))_{d-1-i} \tag{3}$$

is an injection for $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$.

Furthermore, we deduce from the sequence (2) that $\dim_k(I/(I \cap \Theta))_t = h_t^{\mathrm{sd}(\Delta)} - h_t^{\mathrm{sd}(\widetilde{\Delta})}$ for $0 \leq t \leq d$.

In order to compute this difference we determine the change in the h-vector of $\widetilde{\Delta}$ when adding the last facet F_m of the shelling. Let $r_m := \#\mathrm{res}(F_m)$. Proposition 4.3 implies $h_{r_m}^{\Delta} = h_{r_m}^{\widetilde{\Delta}} + 1$ and $h_i^{\Delta} = h_i^{\widetilde{\Delta}}$ for $i \neq r_m$. Using Theorem 4.2 we deduce:

$$\begin{aligned} h_i^{\mathrm{sd}(\Delta)} &= \sum_{r=0}^{d} A(d+1, i, r+1) h_r^{\Delta} \\ &= \sum_{r=0}^{d} A(d+1, i, r+1) h_r^{\widetilde{\Delta}} + A(d+1, i, r_m+1) \\ &= h_i^{\mathrm{sd}(\widetilde{\Delta})} + A(d+1, i, r_m+1). \end{aligned}$$

Thus $\dim_k(I/(I \cap \Theta))_i = A(d+1, i, r_m+1)$ for $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$. From (3) it follows that $A(d+1, i, r_m+1) \leq A(d+1, d-1-i, r_m+1)$.

Take Δ to be the boundary of the d-simplex. Since in this case $h_i^{\Delta} \geq 1$ for $0 \leq i \leq d$, i.e. restriction faces of all possible sizes occur in a shelling of Δ, it follows that $A(d+1, i, r) \leq A(d+1, d-1-i, r)$ for every $1 \leq r \leq d+1$ and $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$. This shows (i).

To show (ii) we use that the injections in (3) induce injections

$$\omega: \ (I/(I \cap \Theta))_i \to (I/(I \cap \Theta))_{i+1}$$

for $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$. Thus, $A(d+1, i, r_m+1) \leq A(d+1, i+1, r_m+1)$. The same reasonning as in (i) shows that $A(d+1, i, r) \leq A(d+1, i+1, r+1)$ for $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$ and $1 \leq r \leq d$. The second part of (ii) follows from the first one using Lemma 4.1. $\square$

Example 4.5. $A(6,2,3) = 60 > 48 = A(6,3,3)$ while $A(6,2,4) = 48 < 60 = A(6,3,4)$. This shows that for d odd $A(d+1, \lfloor \frac{d}{2} \rfloor, r+1)$ may be larger or smaller then $A(d+1, \lceil \frac{d}{2} \rceil, r+1)$.

Recall that a sequence of integers $s = (s_0, \ldots, s_d)$ is called unimodal if there is a $0 \leq j \leq d$ such that $s_0 \leq \ldots \leq s_j \geq \ldots \geq s_d$. We call s_j a *peak* of this sequence and say that it is at *position* j (note that j may not be unique).

Remark 4.6. From [3] it can already be deduced that the sequence $(A(d{+}1,0,r{+}1),\ldots,A(d{+}1,d,r{+}1))$ is unimodal. Applying the linear transformation of Theorem 4.2 to the $(r+1)$st unit vector yields the sequence $(A(d+1,0,r+1),\ldots,A(d+1,d,r+1))$. It then follows from [3, Theorem 3.1, Remark 3.3] that the generating polynomial of this sequence is real-rooted. Since $A(d,i,r+1)\geq 1$ for $i\geq 1$ the sequence $(A(d{+}1,0,r{+}1),\ldots,A(d{+}1,d,r))$ has no internal zeros. Together with the real-rootedness this implies that $(A(d{+}1,0,r{+}1),\ldots,A(d{+}1,d,r))$ is unimodal. However, this argument tells nothing about the position of the peak.

Recall that a regular CW-complex Δ is called a *Boolean cell complex* if for each $A\in\Delta$ the lower interval $[\emptyset,A]:=\{\ B\in\Delta\mid\emptyset\leq_\Delta B\leq_\Delta A\}$ is a Boolean lattice, where $A\leq_\Delta A'$ if A is contained in the closure of A' for $A,A'\in\Delta$. In [3] it was shown that the h-vector of the barycentric subdivision of a Boolean cell complex with non-negative entries is unimodal. What remains open is the location of its peak. Using Corollary 4.4 we can solve this problem:

Corollary 4.7. *Let Δ be a $(d-1)$-dimensional Boolean cell complex with $h_i^\Delta\geq 0$ for $0\leq i\leq d$. Then the peak of $h^{\mathrm{sd}(\Delta)}$ is at position $\frac{d}{2}$ if d is even and at position $\frac{d-1}{2}$ or $\frac{d+1}{2}$ if d is odd. In particular, this assertion holds for CM complexes.*

Proof. Since $h_i^\Delta\geq 0$ for $0\leq i\leq d$, by Theorem 4.2 and Corollary 4.4 (ii) we deduce

$$\begin{aligned} h_j^{\mathrm{sd}(\Delta)} &= \sum_{r=0}^{d} A(d+1,j,r+1)h_r^\Delta \\ &\overset{\text{Corollary } 4.4(ii)}{\leq} \sum_{r=0}^{d} A(d+1,j+1,r+1)h_r^\Delta = h_{j+1}^{\mathrm{sd}(\Delta)} \end{aligned}$$

for $0\leq j\leq\lfloor\frac{d-2}{2}\rfloor$. Thus $h_0^{\mathrm{sd}(\Delta)}\leq h_1^{\mathrm{sd}(\Delta)}\leq\ldots\leq h_{\lfloor\frac{d}{2}\rfloor}^{\mathrm{sd}(\Delta)}$.

Similarly one shows $h_{\lceil\frac{d}{2}\rceil}^{\mathrm{sd}(\Delta)}\geq h_{\lceil\frac{d}{2}\rceil+1}^{\mathrm{sd}(\Delta)}\geq\ldots\geq h_d^{\mathrm{sd}(\Delta)}$, when applying Corollary 4.4 (ii) for $j\geq\lceil\frac{d}{2}\rceil$.

If d is even $\lfloor\frac{d}{2}\rfloor=\lceil\frac{d}{2}\rceil=\frac{d}{2}$ and the peak of $h^{\mathrm{sd}(\Delta)}$ is at position $\frac{d}{2}$. $\square$

Example 4.8. If d is odd, depending on whether $h_{\lfloor\frac{d}{2}\rfloor}^{\mathrm{sd}(\Delta)}\leq h_{\lceil\frac{d}{2}\rceil}^{\mathrm{sd}(\Delta)}$ or vice versa the peak of $h^{\mathrm{sd}(\Delta)}$ is at position $\frac{d-1}{2}$ or $\frac{d+1}{2}$. For example, for $d=3$ let Δ be the 2-skeleton of the 4-simplex. Then $h^\Delta=(1,2,3,4)$ and $h^{\mathrm{sd}(\Delta)}=(1,22,33,4)$, i.e. the peak is at position $\frac{3+1}{2}=2$.

If Δ consists of 2 triangles intersecting along one edge, i.e. $\Delta := \langle\{1,2,3\},\{2,3,4\}\rangle$, then $h^{\Delta} = (1,1,0,0)$ and $h^{\mathrm{sd}(\Delta)} = (1,8,3,0)$. In this case the h-vector peaks at position $\frac{3-1}{2} = 1$.

Using Corollary 4.4 we establish also the following inequalities; compactly summarized later in Corollary 4.10.

Corollary 4.9. (i) *$A(d+1,j,1) \leq A(d+1,j,2) \leq \ldots \leq A(d+1,j,d+1)$ for $\lceil\frac{d+1}{2}\rceil = \lfloor\frac{d+2}{2}\rfloor \leq j \leq d$.*

(ii) *$A(d+1,j,1) \geq A(d+1,j,2) \geq \ldots \geq A(d+1,j,d+1)$ for $0 \leq j \leq \lfloor\frac{d-1}{2}\rfloor$.*

(iii) *$A(d+1,\frac{d}{2},1) \leq A(d+1,\frac{d}{2},2) \leq \ldots \leq A(d+1,\frac{d}{2},\frac{d}{2}+1) \geq A(d+1,\frac{d}{2},\frac{d}{2}+2) \geq \ldots \geq A(d+1,\frac{d}{2},d+1)$ if d is even.*

(iv) *$A(d+1,j,1) = A(d+1,j+1,d+1)$ for $0 \leq j \leq d-1$.*

Proof. To prove (i) we need to show that $A(d+1,j,r) \leq A(d+1,j,r+1)$ for $1 \leq r \leq d$ and $\lfloor\frac{d+2}{2}\rfloor \leq j \leq d$. For $j = d$ this follows from $\{\ \sigma \in S_{d+1} \mid \mathrm{des}(\sigma) = d\} = \{\ (d+1)d\ldots 21\ \}$. Let $C^d_{j,r} := \{\ \sigma \in S_{d+1} \mid \mathrm{des}(\sigma) = j, \sigma(1) = r\}$. Consider the following map:

$$\begin{array}{rcl} \phi^d_{j,r} : \{\sigma \in C^d_{j,r} \mid \sigma(2) \neq r+1\} & \to & \{\ \sigma \in C^d_{j,r+1} \mid \sigma(2) \neq r\} \\ \sigma & \mapsto & (r,r+1)\sigma. \end{array}$$

For $\sigma \in C^d_{j,r}$, if $\mathrm{des}(\sigma) = j$ and $\sigma(2) \neq r+1$ then σ and $(r,r+1)\sigma$ have the same descent set, hence $\mathrm{des}((r,r+1)\sigma) = j$ as well.

As $((r,r+1)\sigma)(1) = r+1$ the function $\phi^d_{j,r}$ is well-defined. Since $(r,r+1)^2 = id$ it follows that $\phi^d_{j,r}$ is invertible and therefore $\#\{\ \sigma \in C^d_{j,r} \mid \sigma(2) \neq r+1\} = \#\{\ \sigma \in C^d_{j,r+1} \mid \sigma(2) \neq r\}$.
If $\sigma \in C^d_{j,r}$ and $\sigma(2) = r+1$, then all of the j descents must occur at position at least 2.

The sequence $\tilde{\sigma} = (r+1)\sigma(3)\ldots\sigma(d+1)$ can be identified with a permutation τ in S_d with $\tau(1) = r$ and vice versa via the order preserving map $[d+1]\setminus\{r\} \to [d]$, hence the descent set is preserved under this identification. Therefore $\#\{\ \sigma \in C^d_{j,r} \mid \sigma(2) = r+1\} = \#\{\ \sigma \in C^{d-1}_{j,r}\} = A(d,j,r)$. On the other hand, if $\sigma \in C^d_{j,r+1}$ and $\sigma(2) = r$ then σ has exactly $j-1$ descents at positions $\{2,\ldots,d\}$. A similar argumentation as before then implies $\#\{\ \sigma \in C^d_{j,r+1} \mid \sigma(2) = r\} = \#\{\ \sigma \in C^{d-1}_{j-1,r}\} = A(d,j-1,r)$.

By Corollary 4.4(ii) it holds that $A(d,j,r) \leq A(d,j-1,r)$ for $d-2 \geq j-1 \geq \lceil\frac{d-1}{2}\rceil$, i.e. $d-1 \geq j \geq \lceil\frac{d+1}{2}\rceil = \lfloor\frac{d+2}{2}\rfloor$. Combining the above, we obtain $A(d+1,j,r) \leq A(d+1,j,r+1)$ for $1 \leq r \leq d$ and $\lfloor\frac{d+2}{2}\rfloor \leq j \leq d-1$, and (i) follows.

(ii) follows directly from (i) and Lemma 4.1.

For the proof of (iii) we only show $A(d+1, \frac{d}{2}, 1) \leq A(d+1, \frac{d}{2}, 2) \leq \ldots \leq A(d+1, \frac{d}{2}, \frac{d}{2}+1)$. The other inequalities in (iii) follow directly from this part by Lemma 4.1. The proof of (i) shows that $\#\{\ \sigma \in C^d_{\frac{d}{2},r} \mid \sigma(2) \neq r+1\} = \#\{\ \sigma \in C^d_{\frac{d}{2},r+1} \mid \sigma(2) \neq r\}$. As in the proof of (i), it remains to prove that $A(d, \frac{d}{2}, r) \leq A(d, \frac{d}{2}-1, r)$ for $1 \leq r \leq \frac{d}{2}$. By Lemma 4.1 it holds that $A(d, \frac{d}{2}, r) = A(d, \frac{d}{2}-1, d+1-r)$. For $1 \leq r \leq \frac{d}{2}$ we have $r \leq d+1-r$ and (ii) then implies $A(d, \frac{d}{2}-1, r) \geq A(d, \frac{d}{2}-1, d+1-r)$ which finishes the proof of (iii).

To show (iv) note that by Lemma 4.1 $A(d+1, j, 1) = A(d+1, d-j, d+1)$. If $\sigma = (d+1)\sigma(2)\ldots\sigma(d+1) \in C^d_{d-j,d+1}$, then the 'reverse' permutation $\widetilde{\sigma} := (d+1)\sigma(d+1)\ldots\sigma(2)$ has a descent at position 1 and whenever there is an ascent in $\sigma(2)\ldots\sigma(d+1)$. Since σ has $d-j-1$ descents at positions $\{2, \ldots, d\}$ this implies $\mathrm{des}(\widetilde{\sigma}) = 1+(d-1)-(d-j-1) = j+1$, i.e. $\widetilde{\sigma} \in C^d_{j+1,d+1}$. We recover σ by repeating this construction and hence $A(d+1, j, 1) = A(d+1, j+1, d+1)$. $\square$

Let $\mathcal{A} := (A(d,i,j))_{i,j}$ be the matrix with entries $A(d,i,j)$ for fixed d. For pairs $(i,j), (i',j')$ we set $(i,j) < (i',j')$ if either $i < i'$ or $i = i'$ and $j > j'$. This defines a total order on the set of pairs (i,j). Using this ordering for the indices of the entries of the matrix we can write the matrix $\mathcal{A}$ as a vector $A(d)$.

From Corollary 4.9 and Lemma 4.1 we immediately get the following.

Corollary 4.10. *The sequence $A(d)$ is unimodal and symmetric for $d \geq 1$. In particular, the peak of $A(d)$ lies in the middle.* $\square$

The numerical results in Corollaries 1.3 and 4.7 suggest that the barycentric subdivision of a Cohen-Macaulay simplicial complex might be weak Lefschetz. Recall that a $(d-1)$-dimensional simplicial complex is called *weak Lefschetz over k* if there exists a maximal l.s.o.p. Θ for $k[\Delta]$ and a degree one element $\omega \in (k[\Delta]/\Theta)_1$ such that the multiplication maps

$$\omega : \ (k[\Delta]/\Theta)_i \ \longrightarrow \ (k[\Delta]/\Theta)_{i+1}$$

have full rank for every i. In particular, in our case this means injections for $0 \leq i < \frac{d}{2}$ and surjections for $\lceil \frac{d}{2} \rceil \leq i \leq \dim \Delta$.

Problem 4.11. Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex and k be an infinite field. Then the barycentric subdivision of Δ is weak Lefschetz over k.

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